International Journal of Theoretical Physics, Vol. 37, No. 5, 1998

Percolation on Inhomogeneous Bethe Lattice and Forest Fire Models

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Received July 30, 1997

The inhomogeneous Bethe lattice (IBL) is defined and studied. It is used to study the random neighbor for forest fire model, and we show that it is more realistic than the Bethe lattice, and gives large probability for the subcritical case

1. BASIC CONCEPTS

Percolation (Stauffer and Aharony, 1992) on the Bethe lattice (Cayley tree) is an important example of an exactly solvable problem. It also has many applications. In all such studies the number of nearest neighbors of any site (the coordination number) is fixed. It is denoted by z, for example, in the study of the immune system (Ahmed and Abdusalam, 1994). Some applications require that z changes from one site to another, i.e., z becomes z_i, where i labels the site. We call this lattice the Inhomogeneous Bethe lattice (IBL).

To evaluate the critical concentration p_c it is noticed that the number of branches outgoing from site i is $z_i - 1$. It is assumed that there is one incoming branch to site i. Hence the average number of open paths from site i is $p(z_i - 1)$. Thus, in order for site i to belong to an infinite cluster, the quantity $p(z_i - 1)$ should exceed unity; therefore the critical concentration is

$$p_c = \max_i \left(\frac{1}{z_i - 1} \right) \tag{1}$$

To evaluate the probability that an occupied site belongs to an infinite cluster P, let site j be the nearest neighbor to site i and let Q_j be the probability that site j does not belong to an infinite cluster along a given branch; then

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0020-7748/98/0500-158 7$15.00/0

$$Q_j = 1 - p + p \prod_{k \neq i} Q_k \tag{2}$$

where p is the probability that a site is occupied and k runs over the set of all nearest neighbors of j other than i. The quantity $p - P$ is the probability that a site will be occupied but does not belong to an infinite cluster; hence

$$p - P = p \prod_{j} Q_j \tag{3}$$

Equations (2) and (3) determine P.

It is to be noticed that when $z_i = z$ the familiar equations for p are regained, i.e.,

$$Q = 1 - p + pQ^{z-1}, \qquad p - P = pQ^z \tag{4}$$

2. FOREST FIRE MODELS

Forest fire models are defined on d-dimensional hypercubic lattices of linear size L. Each lattice site is either empty, a tree, or a burning tree. Most forest fire models can be described by the following simple rules, which are used at each time step in order to update the system in parallel:

1. A burning tree becomes an empty site.
2. Trees grow with probability p from empty sites.
3. A green tree becomes a burning tree with probability $(1 - g)$ if at least one next nearest neighbor is burning.
4. A green tree becomes a burning tree with probability $f \ll 1$ if no nearest neighbor is burning.

Taking $g = 0$ in (3) and neglecting (4), one has the model of Bak *et al.* (1990), which is noncritical and exhibits a steady state which is a succession of fire fronts with fractal dimension $D = 1$ (Grassberger and Kantz, 1991) A modification of this model is obtained by introducing the immunity g ($0 < g \leq 1$) in (3)), which is the probability that a tree is not ignited although one of its neighbors is burning (Drossel and Schwable, 1993). This forest fire model with immune trees exhibits interesting fluctuating percolation behavior. Another model proposed by Drossel and Schwable (1992) takes $g = 0$ in (3) and in the limit $f \ll 1$, $f/p \to 0$ shows self-organized criticality (SOC) behavior in nonconservative systems (Christensen *et al.*, 1993). In a recent variant of the forest fire model, which also exhibits SOC, it is assumed that sparks are dropped at random and if they fall on a tree, the whole cluster of sites connected to it burns (Henley, 1993).

3. MEAN-FIELD THEORY OF THE MODEL

We study in detail the mean-field theory of the model proposed by Drossel and Schwable (1992) or, more precisely, a random neighbor version of the forest fire model: We disregard the lattice geometry and consider instead an ensemble of N sites on which trees may grow. The system evolves according to the dynamical rules defined above, but at each time step every site with a burning tree is assigned Z neighbor sites to which the fire will spread to the extent that there are trees on these sites. This neighbor relationship is oriented: fire may spread only one way through it. Neighbor sites are chosen at random from the ensemble and rechosen anew at every time step. The parameter Z is called the coordination number and is identified with $2d - 1$ when the random neighbor model is used as an approximation to the system on a hypercubic lattice. Let $\rho_e(\tau)$, $\rho_t(\tau)$, and $\rho_f(\tau)$ denote respectively the densities of empty sites, trees, and burning trees at time τ. The mean-field equations for the forest fire model are the rate equations for these densities supplemented with the normalization constraint. For the small values of the lightning rate f that we will consider, we may assume—and later find correct—that the number of burning trees at any given time is small compared to the total number of trees, i.e., $N\rho_f \lesssim N\rho_t$. With this assumption, no tree is ignited by more than one burning tree, and the rate equations read

$$\begin{aligned}
\rho_e(\tau + 1) &= (1 - p)\rho_e(\tau) + \rho_f(\tau) \\
\rho_t(\tau + 1) &= [1 - f - Z\rho_f(\tau)]\rho_t(\tau) + p\rho_e(\tau) \\
\rho_f(\tau + 1) &= [f + Z\rho_f(\tau)]\rho_t(\tau) \\
1 &= \rho_e(\tau) + \rho_t(\tau) + \rho_j(\tau)
\end{aligned} \tag{5}$$

and then the three densities change during one time step according to the following equations:

$$\begin{aligned}
\Delta\rho_e(\tau) &= \rho_f(\tau) - p\rho_e(\tau) \\
\Delta\rho_t(\tau) &= p\rho_e(\tau) - [f + Z\rho_f(\tau)]\rho_t(\tau) \\
\Delta\rho_f(\tau) &= (f + Z\rho_f(\tau))\rho_e(\tau) - \rho_f(\tau)
\end{aligned} \tag{6}$$

The time evolution of these equations has an attractive fixed point. Its existence and stability may be understood as follows: With many trees in the forest, fire will propagate easily, and more trees will disapper than are grown. With few trees in the forest, fire will propagate with difficulty, die out fast, and more trees will grow than disappear. While oscillations between these two situations certainly occur locally in finite dimensions, the mean-field equations have an attractive fixed point, and consequently the system will reach a stationary state, i.e., the three densities do not change in time. Setting $\Delta\rho_t(\tau) = \Delta\rho_e(\tau) = \Delta\rho_f(\tau) = 0$ in equation (6), we obtain

$$\begin{aligned} \rho_f(\tau) &= p\rho_e(\tau) \\ p\rho_e(\tau) &= (f + Z\rho_f(\tau))\rho_t(\tau) \\ \rho_f(\tau) &= (f + Z\rho_f(\tau))\rho_t(\tau) \\ 1 &= \rho_f(\tau) + \rho_t(\tau) + \rho_e(\tau) \end{aligned} \tag{7}$$

Using equation (7), one can easily find the density of trees as

$$\rho_t = \frac{Z + 1 + K \pm \sqrt{(Z-1)^2 + 2(Z+1)K + K^2}}{2Z} \tag{8}$$

where

$$K = \frac{f(1+p)}{p}$$

Only the solution with the minus sign is meaningful, the other giving $\rho_t > 1$, which is impossible ($0 \leq \rho_t \leq 1$).

Expanding ρ_t by Taylor series in K, we get

$$\rho_t = \frac{1}{Z} - \frac{1}{Z^2 - Z} K + O(K^2) \tag{9}$$

The forest fire model is described as a random branching process as follows: One burning tree can ignite form 0 to Z other trees, depending on how many of its Z neighbor sites are occupied by trees. By assumption, the fire is so sparse at any time that no tree is ignited by more than one burning tree. Consequently, a space-time map of a forest fire has the topology of a tree (Bethe lattice), each node representing a burning tree, and branches from such a node representing the spreading of the fire to neighbor sites. Since the fire can spread from one burning tree to any number b of trees between 0 and Z, the average number of trees ignited by one burning tree is given by using the binomial distribution and equation (9) as follows:

$$\begin{aligned} \langle b \rangle &= \sum_{b=0}^{Z} b \binom{Z}{b} \rho_t^b (1 - \rho_t)^{Z-b} \\ &= Z\rho_t \\ &= 1 - \frac{1}{Z-1} K + ZO(K^2) \end{aligned} \tag{10}$$

where, in the first identity, we have used that a randomly chosen site contains a tree with probability ρ_t. Since K is proportional to f, in the limit of $f \to 0$,

$\langle b \rangle \to 1$. In this limit the mean-field theory of the forest fire model will be nothing but a critical branching process—critical because $\langle b \rangle = 1$ means a burning tree on the average ignites exactly one tree. Thus the fire continues forever, on the average. If $K \neq 0$, then $\langle b \rangle < 1$ and the system is subcritical, and a fire will die out in a finite time.

Let the geometry of the model be given by an underlying lattice of finite dimension d. Then the fire will self-interact, resulting in different critical exponents, and the dynamical process might induce correlations between sites. We assume that sites are not correlated, and that the lattice has an average density of trees ρ_t. Then we have a *percolation* problem: A fire will burn exactly the cluster of trees in which it was started, and the known cluster-size distribution of the percolation problem is the mean field estimate for the size distribution for forest fires in finite dimensions. In particular, the known exponents of percolation theory are our mean-field estimates for exponents in the forest fire model.

The density of trees can be derived self-consistently using the knowledge of the exponents from percolation theory: In a statistically stationary state the rate of flow into the system (the rate of growth) equals the rate of flow out of the system (the rate of burning), that is, if $\langle s \rangle$ denotes the average size of a forest fire initiated by lightning, then

$$p\rho_e L^d = \langle s \rangle f \rho_t L^d \tag{11}$$

and using $\rho_e = 1 - \rho_t$, we have

$$\langle s \rangle = \frac{p}{f} \frac{1 - \rho_t}{\rho_t} \propto |p_c - \rho_t|^{-\gamma} \tag{12}$$

since the average cluster in percolation theory diverges with the exponent γ. This gives an estimate of the density of trees as a function of p and f. However, we can also estimate γ using the measured value of ρ_t. For this estimate to be consistent, the correlation length in the percolation problem should be much smaller than the lattice size, so that fluctuations in ρ_t are negligible. Since the fractal dimension of clusters in the percolation problem is smaller than the embeding dimension when $d \geq 2$, we expect that the density fluctuations indeed will be unimportant.

The coordination number Z is identified with $2d - 1$ because a fire cannot propagate backward to a site from which it came in the previous time step when p, the probability per step of growing a new tree, is small as it is here.

Now, we can describe the forest fire model as a random pranching process by using the IBL, i.e., the coordination number Z will be replaced by Z_i, where Z_i labels the coordination number at site i which changes from one site to another. In this description, one burning tree can ignite from 0 to $Z_i - 1$ other trees, depending on how many of its Z_j neighbor sites are occupied by trees; the fire density of trees and the average number of trees ignited by one burning trees are given from equations (3.5) and (3.6) by replacing $Z = Z_i$, and we obtain

$$\rho_t = \frac{1}{Z_i} - \frac{1}{Z_i^2 - Z_i} K + O(K^2) \tag{13}$$

$$\langle b_i \rangle = 1 - \frac{1}{Z_t - 1} K + Z_i O(K^2) \tag{14}$$

the density of trees also can be derived self-consistently by using the knowledge of exponents from percolation theory as

$$\langle s \rangle = \frac{p}{f} \frac{1 - \rho_t}{\rho_t} \propto |p_c - \rho_t|^{-\gamma} \tag{15}$$

where

$$p_c = \max_i \left(\frac{1}{Z_i - 1} \right)$$

We conclude that using the inhomogeneous Bethe lattice to describe the forest fire model is more realistic than the Bethe lattice, and gives large probability for the subcritical case.

REFERENCES

Ahmed, E., and Abdusalam, H. A. (1994). *International Journal of Theoretical Physics*, **33**, 687.

Bak, P., Chen, K., and Tang, C. (1990). *Physics Letters A*, **147**, 297.

Christensen, K., Flyvbjerg, H., and Olami, Z. (1993). *Physical Review Letters*, **71**, 2737.

Clar, S., Drossel, B., and Schwald, F. (1994). *Physical Review E*, **50**, 1009, R2399.

Drossel, B., and Schwable, F. (1992). *Physical Review Letters*, **69**, 1629.

Drossel, B., and Schwable, F. (1993). *Physica A*, **199**, 183.

Grassberger, P., and Kantz, H. (1991). *Journal of Statistical Physics*, **63**, 685.

Stauffer, D., and Aharony, A. (1992). *Introduction to Percolation Theory*, Taylor and Francis, London.